Journal of Statistical Physics, Vol. 38, Nos. 3/4, 1985

On the Asymptotic Behaviour of Spitzer's Model for Evolution of One-Dimensional Point Systems

J. Fritz[1,2]

Received October 25, 1983; revised July 16, 1984

A nearest-neighbor gradient dynamics of one-dimensional infinite particle systems is considered; the model admits a two-parameter family of stationary configurations. Some domains of attraction of stationary configurations are described, and the continuum (hydrodynamical) limit of the system is investigated. It is shown that the mean density of points satisfies a nonlinear diffusion equation in the hydrodynamical limit.

KEY WORDS: Gradient dynamics; local stability; lattice approximation; hydrodynamical limit.

1. INTRODUCTION

We are going to investigate the following gradient dynamics of infinite point systems on the real line $\mathbb{R}$. Configurations of the system will be represented as real sequences $\omega = (\omega_k)_{k \in \mathbf{Z}}$ indexed by the set $\mathbf{Z}$ of integers, i.e., $\omega \in \mathbb{R}^{\mathbf{Z}}$. The evolution law is given by the infinite system

$$\dot{\omega}_k = U'(\omega_{k+1} - \omega_k) - U'(\omega_k - \omega_{k-1}), \qquad k \in \mathbf{Z} \tag{1.1}$$

of ordinary differential equations, where $\dot{\omega}_k = d\omega_k/dt$, and U' denotes the derivative of a strictly convex U: $\mathbb{R} \to \mathbb{R}$. The evolution of the distances $\delta_k = \delta_k(\omega) = \omega_{k+1} - \omega_k$ is governed by

$$\dot{\delta}_k = U'(\delta_{k+1}) + U'(\delta_{k-1}) - 2U'(\delta_k), \qquad k \in \mathbf{Z} \tag{1.2}$$

Notice that this evolution law does depend on the enumeration of the particles; we are assuming that $\omega_{k+1} > \omega_k$ for all $k \in \mathbf{Z}$, at least at the initial moment $t = 0$. Since U' is strictly increasing, $\delta_k(t) = 0$ and $\delta_{k-1}(t) > 0$,

[1] Mathematical Institute, H.A.S., H-1364 Budapest, Pf. 127, Hungary.
[2] Research supported by I.H.E.S., Bures-sur-Yvette, France.

0022-4715/85/0200-0615$04.50/0

$\delta_{k+1}(t) > 0$ imply that $\dot{\delta}_k(t) > 0$, therefore $\omega_{k+1}(0) > \omega_k(0)$ for all $k \in \mathbf{Z}$ results in $\omega_{k+1}(t) > \omega_k(t)$ for all $t > 0$ and $k \in \mathbf{Z}$. This means that (1.1) can really be interpreted as an evolution law for point systems on the line.

Gradient systems like (1.1) have been proposed by Spitzer[1] as traffic models; cf. Ref. 5 with some further references. From a general mathematical point of view, (1.1) seems to be the simplest but not explicitly solvable continuous model which exhibits a hydrodynamical behavior; the barycenter and the density of the particles are the related conserved quantities. Let us remark that gradient dynamics of one-dimensional point systems reduces to (1.1) in the following situation. Consider the system

$$\dot{\omega}_k = -\sum_{j \neq k} V'(\omega_k - \omega_j), \qquad k \in \mathbf{Z} \tag{1.3}$$

with a symmetric pair potential V of finite range, and suppose that $V = U$ on an interval $[a, b]$ such that $2a$ is larger than the radius of interaction of V. Since the property $[\delta_k \in [a, b] \colon k \in \mathbf{Z}]$ is preserved by (1.3), and only nearest neighbors can interact in this case, we see that solutions to (1.3) and to (1.1) coincide if $\delta_k(0) \in [a, b]$ for all $k \in \mathbf{Z}$. In this sense (1.1) describes some small fluctuations around the ground states of V. There is a hope that methods developed for the study of (1.1) can be applied to the related stochastic gradient systems, as well.

Perhaps the most transparent feature of (1.1) is the presence of a two-parameter family of stationary points $\theta(z, w) = (z + kw)_{k \in \mathbf{Z}}$, $z, w \in \mathbb{R}$. Stationary measures of the system are concentrated on the set of such equally spaced configurations; see Refs. 4 and 5. The main purpose of this paper is to investigate the asymptotic behavior of solutions in such situations when the initial distribution is not translation invariant. In the next section some domains of attraction will be specified in terms of certain quadratic fluctuations around the stationary points. The basic result of this kind establishes that for initial configurations satisfying

$$\sup_{r \geqslant 1} r^{-\lambda} \sum_{k=-r}^{r} (\omega_k - \omega_0 - kw)^2 < +\infty \tag{1.4}$$

with $\lambda < 3$, for each $k \in \mathbf{Z}$ we have $\delta_k(t) \to w$ as $t \to +\infty$. Of course, given ω one can find at most one w satisfying (1.4) with $\lambda < 3$. If the initial fluctuations are so small that $\lambda < 3$ in (1.4) then asymptotics of solutions is essentially the same as that we have for the best linear approximation

$$\dot{u}_k = \frac{\sigma}{2}(u_{k-1} + u_{k+1} - 2u_k), \qquad k \in \mathbf{Z} \tag{1.5}$$

where $\sigma = 2U''(w)$. As the level of initial fluctuations exceeds the critical value $\lambda = 3$, a more complex behavior begins to develop; an intuitive picture can be obtained in the hydrodynamical (continuum) limit only.

Since (1.1) is a diffusive gradient system (see Ref. 14), the appropriate rescaling of space and time should be given by the rule $x \to x/h$, $t \to t/h^2$, where the scaling parameter $h > 0$ gives the order of the typical distance of consecutive points. More exactly, the hydrodynamical rescaling of the number of particles means the following. Let φ denote a smooth function with a compact support, and introduce the rescaled counting functional

$$N_h(t, \varphi) = h \sum_{k \in \mathbf{Z}} \varphi(h\omega_k(t/h^2)) \tag{1.6}$$

In Section 3 the family μ_h, $h > 0$ of initial distributions will be prescribed in such a way that

$$\lim_{h \to 0} N_h(t, \varphi) = \int_{-\infty}^{+\infty} \varphi(y)\, \rho(t, y)\, dy \tag{1.7}$$

in probability for all continuous φ with compact support, and for each $t \geqslant 0$. The limiting density $\rho = \rho(t, y)$ will be identified as the weak solution to the nonlinear diffusion equation $\dot{\rho} = -(U'(1/\rho))''$, where $\dot{}$ and $'$ denote temporal and spatial derivatives, respectively. Since $1/\rho$ has appeared on the right of the limiting equation, we need conditions ensuring boundedness of $\rho(0, y)$ and of $1/\rho(0, y)$, as well. A similar diffusion equation was obtained by Rost[8] in the case of independent diffusions of hard rods on the line.

In the first part of the study of the hydrodynamical limit described above, (1.1) and (1.2) will be considered as lattice models, i.e., ω_k will be interpreted as an unbounded spin variable at site $k \in \mathbf{Z}$. The related rescaled quantities read as $Z_h(t, x) = \omega_{[x/h]}(t/h^2)$ and $W_h(t, x) = \delta_{[x/h]}(t/h^2)$, where $[u]$ denotes the integer part of $u \in \mathbb{R}$. In this picture the scaling parameter h is just the macroscopic distance of neighboring lattice sites. It is plain that (1.1) and (1.2) are lattice approximations to the partial differential equations $\dot{z} = U''(0)\, z''$, and to $\dot{w} = (U''(w)\, w')'$, respectively. The lattice approximation picture is very convenient; convergence of these continuum limit procedures will be proven by means of a method of Liapunov functions. Actually, we can prove convergence in a scale of Hilbert norms, whence (1.7) follows by means of an *a priori* bound for the number of particles. Quite recently E. Scacciatelli[15] has shown that W_h converges also in the space of continuous functions. This strong form of the local equilibrium is obtained by means of the representation of solutions of (1.2) in terms of the associated inhomogeneous random walk on $\mathbf{Z}$.

The main results of this paper are formulated in the following two sections, proofs and some more detailed statements are presented in the rest of the paper. I wish to express my thanks to E. Presutti and to E. Scacciatelli for useful discussions and remarks.

2. LOCAL STABILITY OF SOLUTIONS

Throughout this paper we are assuming that U is twice continuously differentiable, and

$$0 < c \leqslant U''(x) \leqslant 1/c < +\infty \tag{2.1}$$

for all $x \in \mathbb{R}$. In some cases the uniform Lipschitz condition

$$|U''(x) - U''(y)| \leqslant C \cdot |x - y| \tag{2.2}$$

will be needed, too. Since the transformation $U(x) \to U(x) - U(0) - x \cdot U'(0)$ does not change (1.1), we may (and do) assume that $U(0) = U'(0) = 0$. Since the right-hand side of (1.1) is uniformly Lipschitz continuous, Hilbert space methods are available to study existence and uniqueness of solutions; see Ref. 10. Indeed, let $\mathbb{N}$ denote the set of positive integers, and define Ω_e as the space of all $w \in \mathbb{R}^{\mathbf{Z}}$ such that $\|\omega\|_r < +\infty$ for each $r \in \mathbb{N}$, where

$$\|\omega\|_r = \left[\sum_{n \in \mathbb{N}} e^{-n} \sum_{k=-nr}^{nr} \omega_k^2 \right]^{1/2} \tag{2.3}$$

Let us remark that Ω_e is just the space of configurations with a subexponential growth, i.e., $\omega \in \Omega_e$ if, and only if for any $\varepsilon > 0$ we have $\lim |\omega_k| \exp(-\varepsilon |k|) = 0$ as $|k| \to +\infty$. Our configuration space Ω_e will be equipped with the uniform structure induced by the sequence $\|\omega\|_r$, $r \in \mathbb{N}$ of Hilbert norms. This makes Ω_e a complete separable metric space; convergence of a sequence in Ω_e means convergence with respect to any of these norms. Let $F(\omega) = (U'(\delta_k) - U'(\delta_{k-1}))_{k \in \mathbf{Z}}$ denote the right-hand side of (1.1) as an element of $\mathbb{R}^{\mathbf{Z}}$. It is easy to see that $F: \Omega_e \to \Omega_e$ and for each $r \in \mathbb{N}$ we have

$$\|F(\omega)\|_r \leqslant K(1 + \|\omega\|_r), \qquad \|F(\omega) - F(\bar{\omega})\|_r \leqslant L \, \|\omega - \bar{\omega}_r\|_r \tag{2.4}$$

with some universal K and L. Therefore the general theory[10] yields existence of a unique solution $\omega(t) = \mathbb{P}^t \omega$ for each initial configuration $\omega \in \Omega_e$ such that $\omega(t)$ is a continuous trajectory in Ω_e, $\omega(0) = \omega$, and each coordinate of $\omega(t)$ satisfies (1.1) for all $t \geqslant 0$. Solutions to (1.2) will be represented as $\delta(t) = \delta(\mathbb{P}^t \omega)$, i.e., $\delta_k(t) = \delta_k(\mathbb{P}^t \omega)$.

The domains of attraction of the stationary configurations will be specified in terms of the following subsets of Ω_e. Let $w \in \mathbb{R}$ and $\lambda > 0$, then

Ω_w^λ is defined as the set of $\omega \in \Omega_e$ satisfying (1.4). It is easy to see that $\omega \in \Omega_w^\lambda$ if, and only, if $\|\omega - \theta(\omega_0, w)\|_r = O(r^{\lambda/2})$. If $\lambda < 3$ and $u \neq w$ then $\Omega_w^\lambda \cap \Omega_u^\lambda = \varnothing$. Indeed, (1.4) and the Cauchy inequality imply that

$$\lim_{n\to\infty} n^{-2} \sum_{k=1}^{n} \sum_{j=-k}^{k-1} \delta_j(\omega) = w \tag{2.5}$$

that is if $\omega \in \Omega_w^\lambda$ and $\lambda < 3$ then w is specified as the second Cesaro mean of the sequence $\delta_k(\omega)$ of increments. We shall show in Section 4 that $\mathbb{P}^t\Omega_w^\lambda \subset \Omega_w^\lambda$ for all $w \in \mathbb{R}$, $t > 0$, and $\lambda > 0$. Let us remark that if ω is a typical configuration of a Poisson process of intensity $1/w$ then $(\omega_n - \omega_0 - nw)^2 = O(n)$, thus $\omega \in \Omega_w^\lambda$ can be expected only if $\lambda \geqslant 2$. If $0 < \lambda < 2$ then elements of Ω_w^λ exhibit a long-range order; thus they are closer to equilibrium than the completely random configurations are.

Theorem 2.6. Let $\lambda < 3$; then $\omega \in \Omega_w^\lambda$ implies $\delta_m(\mathbb{P}^t\omega) \to w$ for each $m \in \mathbf{Z}$ as $t \to +\infty$.

This result will be proven in Section 6, where the rate of convergence is also estimated. The bound we have is essentially the same as the exact rate for the linear approximation (1.5) with completely random initial configurations. In the proof the condition (2.2) is not needed.

For the convergence of the central particle the barycenter (the mean spin) of the initial configuration should be specified. What we actually need is

$$\lim_{n\to\infty} n^{-2} \sum_{k=1}^{n} \sum_{j=-k}^{k} \omega_j = z \tag{2.7}$$

It is easy to show that if $\lambda < 2$ and $\omega \in \Omega_w^\lambda$ then (1.4) implies

$$\lim_{n\to\infty} n^{-2} \sum_{k=1}^{n} \sum_{j=m-k}^{m+k} \omega_j = z + mw \tag{2.8}$$

for each $m \in \mathbf{Z}$. The following theorem is an improved version of the qualitative result of Ref. 6 in two directions. The restriction $w = 0$ is removed, and the level of initial fluctuations is considerably higher.

Theorem 2.9. If $\lambda < 2$ and $\omega \in \Omega_w^\lambda$ then (1.4) implies for each $m \in \mathbf{Z}$ that $\lim(\mathbb{P}^t\omega)_m = z + mw$ as $t \to +\infty$.

The proof of Theorem 2.9 will be given in Section 7, where the rate of convergence is estimated as well. The heart of all proofs is a hierarchy of Liapunov functions introduced in Section 4. The related hierarchy of *a priori* bounds is used then in Sections 6 and 7 to show that (1.1) and (1.2) describe asymptotically negligible perturbations of (1.5), at least if the level of initial fluctuations is kept low enough. Some technical tools of this approach are summarized in Section 5. If (2.7) is dropped but $\lambda < 2$ then the bound on the rate of convergence yields a linear diffusion equation for the mean spin in the continuum limit (see Ref. 6). Related questions are discussed in the next section.

3. RESCALING OF SPACE AND TIME

As soon as the level of initial fluctuations has reached the critical level $\lambda = 3$, the perturbative approach of the proofs of local stability does not work any more. Namely, we obtain that solutions to (1.1) or (1.2) and to (1.5) diverge, i.e., if $\lambda \geqslant 3$ then (1.1) or (1.2) cannot be considered as asymptotically negligible perturbations of (1.5). Nevertheless, an intuitive picture can be obtained if space and time are rescaled according to the rule $x \to x/h$ and $t \to t/h^2$, where $h > 0$ goes to zero. This means that the macroscopic distance x and the macroscopic time t correspond to x/h and t/h^2 in the microscopic picture. This scaling principle admits two different interpretations. In the lattice approximation approach (continuum limit) Z_h (or W_h) approximates a continuous function as h goes to zero. Here the lattice site $k = [x/h]$ corresponds to the macroscopic position $x \in \mathbb{R}$. In the point field picture W_h is expected to converge to a proper limit, and not the label k but the actual position ω_k of particles should be rescaled. First we investigate the problem of lattice approximation; the hydrodynamical limit will be based on results in this direction. We are interected in the asymptotics of the following two processes:

$$Z_h(t, x, \omega) = (\mathbb{P}^{t/h^2}\omega)_{[x/h]} \tag{3.1}$$

and

$$\begin{aligned} W_h(t, x, \omega) &= Z_h(t, x+h, \omega) - Z_h(t, x, \omega) \\ &= \delta_{[x/h]}(\mathbb{P}^{t/h^2}\omega) \end{aligned} \tag{3.2}$$

where $h > 0$, $t \geqslant 0$, $x \in \mathbb{R}$, $\omega \in \Omega_e$ and $[u]$ denotes the integer part of $u \in \mathbb{R}$. We prefer the lattice approximation approach, it seems to be more straightforward in both cases than the point process picture.

Suppose now that we are given a family μ_h, $0 < h \leqslant 1$ of probability measures on Ω_e; thus Z_h and W_h will be considered as stochastic processes with values in a space $\mathbb{H}_e$ of real functions defined as follows. If $u: \mathbb{R} \to \mathbb{R}$ is locally integrable then

$$\|u\|_r = \left[\sum_{n=1}^{\infty} e^{-n} \int_{-nr}^{nr} u^2(x)\, dx \right]^{1/2} \tag{3.3}$$

is well defined for $r > 0$; actually it is a Hilbert norm. Analogously to Ω_e let $\mathbb{H}_e$ be the space of all locally integrable functions $u: \mathbb{R} \to \mathbb{R}$ such that $\|u\|_r$ is finite for each $r \in \mathbb{N}$. Let us remark that the mapping $\omega \to u$ given by $u(x) = \omega_{[x/h]}$ is a natural embedding of Ω_e into $\mathbb{H}_e$ for each $h > 0$. On the other hand, if $h > 0$ and $u \in \mathbb{H}_e$ then the step function $u_h(x) = u(h[x/h])$ can be identified with a configuration $\omega \in \Omega_e$ by $\omega_k = u_h(kh) = u(kh)$. Let $\mathbb{H}_e$ be equipped with the uniform structure induced by the sequence $\|\,.\,\|_r$, $r \in \mathbb{N}$ of norms, i.e., $\lim u_n = u$ in $\mathbb{H}_e$ means that we have $\lim \|u_n - u\|_r = 0$ for each $r \in \mathbb{N}$. The space of absolutely continuous $u \in \mathbb{H}_e$ with $u' \in \mathbb{H}_e$ will be denoted by $\mathbb{H}_e^1$, it is natural to define $\lim u_n = u$ in $\mathbb{H}_e^1$ by $\lim \|u_n - u\|_r + \|u_n' - u'\|_r = 0$ for each $r \in \mathbb{N}$. It is easy to see that both $\mathbb{H}_e$ and $\mathbb{H}_e^1$ are complete metrizable spaces. The spaces of continuous mappings of $[0, +\infty)$ into $\mathbb{H}_e$ and $\mathbb{H}_e^1$ will be denoted by $\mathbb{C}(\mathbb{R}_+, \mathbb{H}_e)$ and by $\mathbb{C}(\mathbb{R}_+, \mathbb{H}_e^1)$, respectively. Finally, let $\mathbb{H}_e^2$ be the set of such $u \in \mathbb{H}_e^1$ that u' is absolutely continuous and $u'' \in \mathbb{H}_e$, the spaces of real functions $\varphi: \mathbb{R} \to \mathbb{R}$ with compact support and having continuous first and second derivatives will be denoted by $\mathbb{C}_0^1(\mathbb{R})$ and by $\mathbb{C}_0^2(\mathbb{R})$, respectively.

Now we are in a position to describe the continuum limit of (1.1). Since W_h converges to zero if Z_h has a continuous limit as $h \to 0$, a linear diffusion equation will be obtained.

Theorem 3.4. Let $z_0 \in \mathbb{H}_e^1$ and suppose that for each $r \in \mathbb{N}$ we have

$$\lim_{h \to 0} \int_{\Omega_e} \|Z_h(0, ., \omega) - z_0\|_r^2 \, \mu_h(d\omega) = 0$$

If $U''(0) = 1/2$ then

$$\lim_{h \to 0} \sup_{0 \leqslant t \leqslant T} \int_{\Omega_e} \|Z_h(t, ., \omega) - z(t, .)\|_r^2 \, \mu_h(d\omega) = 0$$

holds for each $T > 0$ and $r \in \mathbb{N}$, where

$$z(t, x) = (2\pi t)^{-1/2} \int_{-\infty}^{+\infty} \exp[-(x-y)^2/2t] \, z_0(y) \, dy$$

Theorem 3.4 will be derived from the *a priori* bounds of Section 4 in Section 8. If W_h has a proper limit as $h \to 0$ then a nonlinear diffusion equation is expected. The weak forms of the underlying equations are of fundamental importance for understanding this limiting procedure.

Let $h \neq 0$, then for functions of a spatial variable we define the operators ∇_h, ∇_h^*, and Δ_h by

$$\nabla_h \varphi(x) = \frac{1}{h} [\varphi(x+h) - \varphi(x)] \tag{3.5}$$

and $\nabla_h^* = \nabla_{-h}$, $\Delta_h = \nabla_h \nabla_h^*$, i.e.,

$$\nabla_h^* \varphi(x) = \frac{1}{h}\,[\varphi(x) - \varphi(x-h)] \tag{3.6}$$

$$\Delta_h \varphi(x) = h^{-2}[\varphi(x+h) + \varphi(x-h) - 2\varphi(x)] \tag{3.7}$$

Observe now that (1.2) turns into $\dot{W}_h = \Delta_h U'(W_h)$, and

$$h \sum_{k=-\infty}^{\infty} \varphi(kh)\, \delta_k(\mathbb{P}^{t/h^2}\omega) = \int_{-\infty}^{+\infty} \varphi_h(x)\, W_h(t, x, \omega)\, dx$$

holds with $\varphi_h(x) = \varphi(h[x/h])$ if φ is of compact support, thus

$$\begin{aligned}
\int_{-\infty}^{+\infty} \varphi_h(x)\, W_h(t, x, \omega)\, dx &= \int_{-\infty}^{+\infty} \varphi_h(x)\, W_h(0, x, \omega)\, dx \\
&\quad + \int_0^t \int_{-\infty}^{+\infty} \varphi_h(x)\, \Delta_h U'(W_h(s, x, \omega))\, dx\, ds \\
&= \int_{-\infty}^{+\infty} \varphi_h(x)\, W_h(0, x, \omega)\, dx \\
&\quad + \int_0^t \int_{-\infty}^{+\infty} (\Delta_h \varphi_h(x))\, U'(W_h(s, x, \omega))\, dx\, ds
\end{aligned} \tag{3.8}$$

whence the weak form

$$\begin{aligned}
\int_{-\infty}^{+\infty} \varphi(x)\, w(t, x)\, dx &= \int_{-\infty}^{+\infty} \varphi(x)\, w(0, x)\, dx \\
&\quad + \int_0^t \int_{-\infty}^{+\infty} \varphi''(x)\, U'(w(s, x))\, dx\, ds
\end{aligned} \tag{3.9}$$

of $\dot{w} = (U''(w)\, w')'$ follows for $\varphi \in \mathbb{C}_0^2(\mathbb{R})$ by a formal limiting procedure. Similarily, if $Y_h(t, x, \omega) = hZ_h(t, x, \omega)$ then $W_h = \nabla_h Y_h$ and $\dot{Y}_h = \nabla_h^* U'(\nabla_h Y_h)$; thus for $\varphi \in \mathbb{C}_0^1(\mathbb{R})$ we obtain that

$$\begin{aligned}
\int_{-\infty}^{+\infty} \varphi_h(x)\, Y_h(t, x, \omega)\, dx &= \int_{-\infty}^{+\infty} \varphi_h(x)\, Y_h(0, x, \omega)\, dx \\
&\quad + \int_0^t \int_{-\infty}^{+\infty} \varphi_h(x)\, \nabla_h^* U'(\nabla_h Y_h(s, x, \omega))\, dx\, ds \\
&= \int_{-\infty}^{+\infty} \varphi_h(x)\, Y_h(0, x, \omega)\, dx \\
&\quad - \int_0^t \int_{-\infty}^{+\infty} (\nabla_h \varphi_h(x))\, U'(\nabla_h Y_h(s, x, \omega))\, dx\, ds
\end{aligned} \tag{3.10}$$

whence the weak form of $\dot{y} = U''(y')\,y''$, namely,

$$\int_{-\infty}^{+\infty} \varphi(x)\,y(t,x)\,dx = \int_{-\infty}^{+\infty} \varphi(x)\,y(0,x)\,dx - \int_0^t \int_{-\infty}^{+\infty} \varphi'(x)\,U'(y'(s,x))\,dx\,ds \tag{3.11}$$

follows by a formal argument; (3.9) and (3.11) are connected by $w(t,x) = y'(t,x)$.

Theorem 3.12. Let $y_0 \in \mathbb{H}_e^2$ and suppose for each $r \in \mathbb{N}$ that

$$\lim_{h\to 0} \int_{\Omega_e} \| Y_h(0,.,\omega) - y_0 \|_r^2\, \mu_h(d\omega) = 0$$

then for any $r \in \mathbb{N}$ and $T > 0$ we have

$$\lim_{h\to 0} \sup_{0\leqslant t\leqslant T} \int_{\Omega_e} \| Y_h(t,.,\omega) - y(t,x) \|_r^2\, \mu_h(d\omega) + \lim_{h\to 0} \int_0^T \int_{\Omega_e} \| W_h(t,.,\omega) - y'(t,.) \|_r^2\, \mu_h(d\omega)\,dt = 0$$

where $y \in \mathbb{C}(\mathbb{R}_+, \mathbb{H}_e^1)$ and $y(t,.) \in \mathbb{H}_e^2$ for each $t \geqslant 0$. The limit y is specified by the property that it satisfies (3.11) with initial condition $y(0,.) = y_0$; this Cauchy problem has a unique solution in $\mathbb{C}(\mathbb{R}_+, \mathbb{H}_e^1)$.

The proof of Theorem 3.12 will be given in Section 9. This result can be extended to all dimensions with a slight modification. Since our *a priori* bounds imply existence of weak derivatives only, in the multivariate case $\mathbb{H}_e^1$ should be defined as the space of $u \in \mathbb{H}_e$ with $\mathbb{H}_e$-valued weak derivatives of first order.

A reformulation of Theorem 3.12 for one-dimensional point systems is not quite immediate, because not the positions but the indices of particles have been rescaled here. Suppose now that $\delta_k(\omega) > 0$ μ_h-a.e. for each $h > 0$, and consider the rescaled counting functional

$$N_h(t,\varphi,\omega) = h \sum_{k\in\mathbf{Z}} \varphi[h(\mathbb{P}^{t/h^2}\omega)_k] = \int_{-\infty}^{+\infty} \varphi(Y_h(t,x,\omega))\,dx \tag{3.13}$$

provided that it makes sense. Under conditions of Theorem 3.12 we expect that $N_h(t,\varphi,\omega)$ converges to

$$N(t,\varphi) = \int_{-\infty}^{+\infty} \varphi(y(t,x)\,dx \tag{3.14}$$

as $h \to 0$, furthermore for $\varphi \in \mathbb{C}_0^2(\mathbb{R})$ we have

$$N(t, \varphi) = N(0, \varphi) + \int_0^t \int_{-\infty}^{+\infty} \varphi'(y(s, x))(U'(y'(s, x)))' \, dx \, ds$$

$$= N(0, \varphi) - \int_0^t \int_{-\infty}^{+\infty} \varphi''(y(s, x)) \, y'(s, x) \, U'(y'(s, x)) \, dx \, ds \tag{3.15}$$

We shall show that strict monotonicity of y_0 implies that of $y(t, .)$ for all $t > 0$; in this case let $n(t, y)$ denote the inverse function of $y(t, .)$, and set $\rho(t, y) = 1/y'(t, n(t, y))$. In view of the definition of ρ we have

$$N(t, \varphi) = \int_{-\infty}^{+\infty} \varphi(y) \, \rho(t, y) \, dy \tag{3.16}$$

which suggests that the limiting density ρ satisfies the weak form

$$N(t, \varphi) = N(0, \varphi) - \int_0^t \int_{-\infty}^{+\infty} \varphi''(y) \, U'(1/\rho(s, y)) \, dy \, ds \tag{3.17}$$

of $\dot{\rho} = -(U'(1/\rho))''$. To prove these assertions we need a property of local finiteness for the initial configuration. The condition $W_h > 0$ plays an important role in the proof.

Theorem 3.18. Suppose all conditions of Theorem 3.12 and let $\delta_k(\omega) > 0$ μ_h-a.s. for all $k \in \mathbf{Z}$ and $h > 0$; moreover let

$$\lim_{r \to +\infty} \sup_{h > 0} \mu_h[Y_h(0, -r, .) < -b, Y_h(0, r, .) > b] = 1$$

for each $b > 0$. If y_0 is strictly increasing then ρ is locally integrable, $N(t, \varphi)$ is given by (3.16), and (3.17) holds for any $\varphi \in \mathbb{C}_0^2(\mathbb{R})$. Finally, if $\varphi: \mathbb{R} \to \mathbb{R}$ is continuous with a compact support, then for each $\varepsilon > 0$ and $T > 0$

$$\lim_{h \to 0} \sup_{t \leqslant T} \mu_h[|N_h(t, \varphi, .) - N(t, \varphi)| > \varepsilon] = 0$$

The hydrodynamical limit described in Theorem 3.18 will be derived as a consequence of Theorem 3.12 in Section 10. Conditions of this result are somewhat unusual. It has not been assumed that the expectation of $N_h(t, \varphi, \omega)$ is finite, but we need conditions ensuring the positivity of ρ; cf. the condition of Theorem 3.12. Of course, (1.1) is strange as an evolution law for point systems because the interaction is attractive with an infinite radius, and the strength of the interaction increases with the distance of consecutive points. It is quite possible that if we allow dilute initial data, then the limiting system develops some singularities.

4. THE HIERARCHY OF LIAPUNOV FUNCTIONS

Additive Liapunov functions as $\sum \omega_k^2$, $\sum U(\delta_k)$ and $\sum [U'(\delta_k) - U'(\delta_{k-1})]^2$ are known to have nice contraction properties in the case of finite gradient systems as well as in translation invariant situations; see Refs. 1, 2, 4, 6, and 11. To exploit these properties in the case of infinite systems with non-translation-invariant initial distributions we must control the production (dissipation) and the spatial flow of these extensive quantities. This program will be realized in the lattice picture by means of the following cut-off function $f(x, r)$ interpreted as a smooth version of the indicator function of the interval $[-r, r]$. This cut-off is defined as follows; cf. Ref. 6. Let $g: \mathbb{R} \to (0, 1)$ be a continuously differentiable nonincreasing function such that $g(u) = e^{1-u}$ if $u \geqslant 2$, $g(u) = 5/2e$ if $u \leqslant 1$, and g is concave if $u < 2$. Notice that $0 \leqslant -g'(u) \leqslant g(u) \leqslant e^{1-u}$ and $g(u) \geqslant g(1)\, e^{1-|u|}$ hold for all $u \in \mathbb{R}$. The cut-off function $f: \mathbb{R} \times [1, +\infty) \to (0, 1)$ is now defined as

$$f(x, r) = \int_{-\infty}^{+\infty} g(|x-y|/r)\, e^{-2|y|}\, dy \tag{4.1}$$

The proofs of the *a priori* bounds are all based on the following elementary properties of f, among which (4.5) is the crucial one. It is essential that r is bounded away from zero; that is why $r \geqslant 1$ is always assumed. Since $g(|x-y|/r) \leqslant \exp(1 + |y| - |x|/r)$ and $g(|x-y|/r) \geqslant g(|y| + |x|/r) \geqslant g(1)$ $\exp(1 - |y| - |x|/r)$, we have

$$\exp(-|x|/r) \leqslant f(x, r) \leqslant 2 \exp(1 - |x|/r) \tag{4.2}$$

thus an easy calculation results in

$$\tfrac{1}{2} \|u\|_r^2 \leqslant \int_{-\infty}^{+\infty} f(x, r)\, u^2(x)\, dx \leqslant 12\, \|u\|_r^2 \tag{4.3}$$

for $u \in \mathbb{H}_e$. Let f' and ∇f denote the partial derivatives of f with respect to r and x. Since

$$f'(x, r) = -r^2 \int_{-\infty}^{+\infty} g'(|x-y|/r)\, |x-y|\, e^{-2|y|}\, dy$$

and

$$|\nabla f(x, r)| \leqslant -\frac{1}{r} \int_{-\infty}^{+\infty} g'(|x-y|/r)\, e^{-2|y|}\, dy$$

but $-g'(u) \leqslant g(u)$ for all $u \in \mathbb{R}$, while $g'(u) = 0$ if $u \leqslant 1$; consequently,

$$|\nabla f(x, r)| \leqslant \min[f'(x, r), r^{-1} f(x, r)] \tag{4.4}$$

Finally, as

$$f(x, r) = \int_{-\infty}^{+\infty} g(|y|/r)\, e^{-2|x-y|}\, dy$$

and

$$f'(x, r) = -r^{-2} \int_{-\infty}^{+\infty} g'(|y|/r)\, |y|\, e^{-2|x-y|}\, dy$$

we obtain for any $x, z \in \mathbb{R}$ that

$$f(x, r) \leqslant f(z, r)\, e^{2|x-z|} \qquad \text{and} \qquad f'(x, r) \leqslant f'(z, r)\, e^{2|x-z|}$$

thus for $0 < h \leqslant 1$ we have

$$\begin{aligned}[\nabla_h f(x, r)]^2 &= h^{-2}[f(x+h, r) - f(x, r)]^2 \\ &\leqslant \frac{80}{r} \min[f(x, r), f(x+h, r)] \min[f'(x, r), f'(x+h, r)]\end{aligned} \tag{4.5}$$

Now we are in a position to introduce a hierarchy P, Q, R, S of Liapunov functions each of which is subordinated to the previous one. Let $u \in \mathbb{C}(\mathbb{R}_+, \mathbb{H}_e)$, then the following functionals are well defined for $r \geqslant 1$, $t \geqslant 0$, and $h > 0$; see (4.3) and (2.1). Since all statements of Section 3 concern the case when h goes to zero, while $h = 1$ in Section 2, we may assume that $h \leqslant 1$. Introduce first

$$v_h(t, x) = \nabla_h u(t, x) = \frac{1}{h}[u(t, x+h) - u(t, x)] \tag{4.6}$$

and $F_h(u) = \nabla_h^* U'(v_h)$, i.e.,

$$F_h(u)(t, x) = \frac{1}{h}[U'(v_h(t, x)) - U'(v_h(t, x-h))] \tag{4.7}$$

and consider

$$P(u, t, r) = \int_{-\infty}^{+\infty} f(x, r)\, u^2(t, x)\, dx \tag{4.8}$$

$$Q_h(u, t, r) = \int_{-\infty}^{+\infty} f(x, r)\, v_h^2(t, x)\, dx \tag{4.9}$$

$$R_h(u, t, r) = \int_{-\infty}^{+\infty} f(x, r)[F_h(u)(t, x)]^2\, dx \tag{4.10}$$

$$S_h(u, t, r) = \int_{-\infty}^{+\infty} f(x, r)[\nabla_h F_h(u)(t, x)]^2\, dx \tag{4.11}$$

We are interested in the evolution of these quantities along solutions to $\dot{u}=F_h(u)$; then $\dot{v}_h=\nabla_h F_h(u)=\Delta_h U'(v_h)$. Since F_h satisfies $\|F_h(u)\|_r \leqslant K_h \|u\|_r$ and $\|F_h(u)-F_h(\bar{u})\|_r \leqslant L_h \|u-\bar{u}\|_r$ for each $r \in \mathbb{N}$, given $u(0,.) \in \mathbb{H}_e$, there is exactly one $u \in \mathbb{C}(\mathbb{R}_+, \mathbb{H}_e)$ such that $\dot{u}=F_h(u)$. In the forthcoming calculations the following identities will be of fundamental importance. Let $\mathbf{T}_h$ denote the shift of a function φ of a spatial variable x, i.e., $\mathbf{T}_h\varphi(x)=\varphi(x+h)$, then

$$2\nabla_h(f\varphi)=(f+\mathbf{T}_h f)\,\nabla_h\varphi+(\nabla_h f)(\varphi+\mathbf{T}_h\varphi) \tag{4.12}$$

and

$$\int_{-\infty}^{+\infty} f\nabla_h^*\varphi\,dx=-\int_{-\infty}^{+\infty}(\nabla_h f)\varphi\,dx \tag{4.13}$$

Proposition 4.14. Let $u, \bar{u} \in \mathbb{C}(\mathbb{R}_+, \mathbb{H}_e)$ be solutions to $\dot{u}=F_h(u)$, and set $\rho(s)=[r^2+Mt-Ms]^{1/2}$ for $0 \leqslant s \leqslant t$ and $r \geqslant 1$, where $M=160/c$ with c as in (2.1); then

$$P(u-\bar{u},t,r)+c\int_0^t Q_h(u-\bar{u},s,\rho(s))\,ds \leqslant P(u-\bar{u},0,(r^2+Mt)^{1/2})$$

for all $r \geqslant 1$ and $t \geqslant 0$.

Proof. Differentiating with respect to time we obtain

$$\begin{aligned}\dot{P}(u-\bar{u},t,r)&=2\int_{-\infty}^{+\infty} f(x,r)[u(t,x)-\bar{u}(t,x)][F_h(u)-F_h(\bar{u})]\,dx\\&=-2\int_{-\infty}^{+\infty}[\nabla_h(fu-f\bar{u})][U'(v_h)-U'(\bar{v}_h)]\,dx\\&=-\int_{-\infty}^{+\infty}(f+\mathbf{T}_h f)(v_h-\bar{v}_h)[U'(v_h)-U'(\bar{v}_h)]\,dx\\&\quad-\int_{-\infty}^{+\infty}(\nabla_h f)(u-\bar{u}+\mathbf{T}_h u-\mathbf{T}_h\bar{u})[U'(v_h)-U'(\bar{v}_h)]\,dx\end{aligned}$$

Observe now that (2.1) implies

$$c(x-y)^2 \leqslant (x-y)[U'(x)-U'(y)]$$

and

$$[U'(x)-U'(y)]^2 \leqslant \frac{1}{c}(x-y)[U'(x)-U'(y)]$$

thus using $py - qy^2 \leqslant p^2/4q$ and (4.5) we obtain that

$$\begin{aligned}
&\dot{P}(u-\bar{u},t,r)+cQ_h(u-\bar{u},t,r)\\
&\quad\leqslant \frac{1}{4c}\int_{-\infty}^{+\infty}(\nabla_h f)^2(\mathbf{T}_h f)^{-1}(u-\bar{u}+\mathbf{T}_h u-\mathbf{T}_h\bar{u})^2\,dx\\
&\quad\leqslant \frac{40}{cr}\int_{-\infty}^{+\infty}\min[f',\mathbf{T}_h f'][(u-\bar{u})^2+(\mathbf{T}_h u-\mathbf{T}_h\bar{u})^2]\,dx\\
&\quad\leqslant \frac{80}{cr}\int_{-\infty}^{+\infty}f'(x,r)[u(x,t)-\bar{u}(t,x)]^2\,dx
\end{aligned}$$

that is,

$$\dot{P}(u-\bar{u},t,r)+cQ_h(u-\bar{u},t,r)\leqslant \frac{M}{2\cdot r}P'(u-\bar{u},t,r) \tag{4.15}$$

where P' denotes the partial derivative of P with respect to r. Therefore, if we let r depend on time in such a way that $\dot{r}+M/2r\leqslant 0$, then P turns to be a decreasing function of time, thus putting $r=\rho(s)$ we obtain the statement. ■

Contraction properties of Q_h are somewhat weaker.

Proposition 4.16. Let $u\in\mathbb{C}(\mathbb{R}_+,\mathbb{H}_e)$ be a solution of $\dot{u}=F_h(u)$, and choose a stationary solution $\theta(x)=z+wx$, then

$$Q_h(u-\theta,t,r)+c\int_0^t R_h(u,s,\rho(s))\,ds\leqslant Q_h(u-\theta,0,(r^2+Mt)^{1/2})$$

where ρ, c, M are the same as in Proposition 4.14.

Proof. In a similar way as above we obtain that

$$\begin{aligned}
&\dot{Q}_h(u-\theta,t,r)\\
&\quad=2\int_{-\infty}^{+\infty}f(x,r)(v_h(t,x)-w)\,\nabla_h F_h(u)\,dx\\
&\quad=-2\int_{-\infty}^{+\infty}[\nabla_h(fv_h-fw)]\,F_h(\mathbf{T}_h u)\,dx\\
&\quad=-\int_{-\infty}^{+\infty}(f+\mathbf{T}_h f)(\nabla_h v_h)[\nabla_h U'(v_h)]\,dx\\
&\qquad-\int_{-\infty}^{+\infty}(\nabla_h f)(v_h+\mathbf{T}_h v_h-2w)[\nabla_h U'(v_h)]\,dx
\end{aligned}$$

$$\leqslant -cR_h(u,t,r) + \frac{1}{4\cdot c}\int_{-\infty}^{+\infty} (\nabla_h f)^2 \frac{1}{f}(v_h + \mathbf{T}_h v_h - 2w)^2\,dx$$

$$\leqslant -cR_h(u,t,r) + \frac{80}{cr}\int_{-\infty}^{+\infty} f'(x,r)[v_h(t,x) - w]^2\,dx$$

$$= -cR_h(u,t,r) + \frac{M}{2\cdot r} Q_h'(u-\theta,t,r)$$

which proves the statement. ■

Finally, for R_h we have the following.

Proposition 4.17. If $u \in \mathbb{C}(\mathbb{R}_+, \mathbb{H}_e)$ satisfies $\dot{u} = F_h(u)$, then

$$R_h(u,t,r) + c\int_0^t S_h(u,s,\rho(s))\,ds \leqslant R_h(u-\theta, 0, (r^2+Mt)^{1/2}),$$

where c, ρ, M are as in Proposition 4.14.

Proof. In a similar way as above we obtain that

$$\dot{R}_h(u,t,r) = 2\int_{-\infty}^{+\infty} fF_h(u)\,\nabla_h^*[U''(v_h)\,\nabla_h F_h(u)]\,dx$$

$$= -2\int_{-\infty}^{+\infty} [\nabla_h(fF_h(u))]\,U''(v_h)\,\nabla_h F_h(u)\,dx$$

$$= -\int_{-\infty}^{+\infty} (f + \mathbf{T}_h f)\,U''(v_h)[\nabla_h F_h(u)]^2\,dx$$

$$= -\int_{-\infty}^{+\infty} (\nabla_h f)\,U''(v_h)[F_h(u) + \mathbf{T}_h F_h(u)]\,\nabla_h F_h(u)\,dx$$

$$- cS_h(u,t,r) + \frac{1}{4\cdot c}\int_{-\infty}^{+\infty} (\nabla_h f)^2(\mathbf{T}_h f)^{-1}$$

$$\times [F_h(u) + \mathbf{T}_h F_h(u)]^2\,dx$$

consequently by (4.5),

$$\dot{R}_h(u,t,r) + cS_h(u,t,r) \leqslant \frac{M}{2\cdot r} R_h'(u,t,r)$$

which completes the proof in the same way as above. ■

Since $Q_h(u-\theta, s, \rho(s))$ is a decreasing function of s, Proposition 4.14 yields

$$
\begin{aligned}
&(t+1)\, Q_h(u-\theta, t, r) \\
&\qquad \leqslant \frac{1}{c} P(u-\theta, 0, (r^2+Mt)^{1/2}) + Q_h(u-\theta, 0, (r^2+Mt)^{1/2}) \qquad (4.18)
\end{aligned}
$$

On the other hand, $R_h(u, s, \rho(s))$ is also decreasing and Proposition 4.15 implies

$$
2(t-s+1)\, R_h(u, t, r) \leqslant \frac{2}{c} Q_h(u-\theta, s, \rho(s)) + 2R_h(u, s, \rho(s))
$$

thus integrating over $0 < s < t$ and adding $R_h(u, t, r) \leqslant R_h(u, 0, \rho(0))$ we obtain

$$
\begin{aligned}
&(t+1)^2\, R_h(u, t, r) \\
&\qquad \leqslant 2c^{-2} P(u-\theta, 0, (r^2+Mt)^{1/2}) \\
&\qquad\quad + \frac{2}{c} Q_h(u-\theta, 0, (r^2+Mt)^{1/2}) + R_h(u, 0, (r^2+Mt)^{1/2}) \qquad (4.19)
\end{aligned}
$$

Let us remark that $R_h \leqslant 2(ch)^{-2}\, Q_h$ and $Q_h \leqslant 2h^{-2}P$, thus if $h > 0$ is fixed then we have bounds for Q_h and R_h at $t > 0$ in terms of P at $t = 0$. The condition (2.2) has not been used in this section.

5. SOME PROPERTIES OF BESSEL FUNCTIONS

In the following two sections (1.2) and (1.1) will be rewritten as

$$
\dot{v}_k(t) = \frac{\sigma}{2} [v_{k-1}(t) + v_{k+1}(t) - 2v_k(t)] + c_k(t), \qquad k \in \mathbf{Z} \qquad (5.1)
$$

where $\sigma = 2U''(w)$ and $c = (c_k(t))_{k \in \mathbf{Z}}$ is a continuous trajectory in Ω_e. If $v(0) \in \Omega_e$ then iterating the linear part of the right-hand side of (5.1) we obtain that

$$
v_m(t) = \sum_{n \in \mathbf{Z}} I_n(\sigma t)\, v_{m-n}(0) + \int_0^t \sum_{n \in \mathbf{Z}} I_n(\sigma t - \sigma s)\, c_{m-n}(s)\, ds \quad (5.2)
$$

where

$$
I_n(t) = \frac{1}{\pi} \int_0^{\pi} \exp[t(\cos x - 1)](\cos nx)\, dx, \qquad n \in \mathbf{Z} \qquad (5.3)
$$

are the Bessel functions of first order with imaginary argument. Let us remark that $I_n(t) = \text{Prob}[X_t = n \mid X_0 = 0]$ if X_t is the standard symmetric random walk with continuous time on $\mathbf{Z}$. Since $I_n \geqslant 0$ and for any $s \in \mathbb{R}$ we have

$$\sum_{n \in \mathbf{Z}} I_n(t)\, e^{sn} = \exp[t(\cosh s - 1)]$$

it is easy to verify (5.2). First we summarize some elementary properties of I_n. It follows directly from (5.3) that $I_n(t)$ is a symmetric probability distribution for each $t \geqslant 0$, i.e., $I_n(t) \geqslant 0$ and $I_n(t) = I_{-n}(t)$ for all $n \in \mathbf{Z}$; furthermore

$$\sum_{n \in \mathbf{Z}} I_n(t) = 1 \tag{5.4}$$

It will be very important that $I_{n+1}(t) \leqslant I_n(t)$ if $n \geqslant 0$; see (3.5) in Ref. 6. Thus the trigonometric identity

$$I_{n-1}(t) - I_{n+1}(t) = \frac{2 \cdot n}{t} I_n(t) \tag{5.5}$$

implies for $n \geqslant 0$ and $t \geqslant 0$ that

$$I_n(t) - I_{n+1}(t) \leqslant \frac{4n+4}{1+t} I_n(t) \tag{5.6}$$

Finally, if $t \geqslant 0$ and $\rho \geqslant 0$ then for $0 \leqslant \lambda \leqslant 4$ we have

$$\sum_{n=0}^{\infty} [(n+1)^2 + \rho]^{\lambda/2} I_n(t) \leqslant 2(1 + t + \rho)^{\lambda/2} \tag{5.7}$$

Indeed, the second derivative of $\exp[t(\cosh s - 1)]$ gives

$$\sum_{n \in \mathbf{Z}} n^2 I_n(t) = t$$

while the fourth one yields

$$\sum_{n \in \mathbf{Z}} n^4 I_n(t) = t + 3t^2$$

whence

$$\begin{aligned}
&\sum_{n \in \mathbf{Z}} [(n+1)^2 + \rho]^2 I_n(t) \\
&\qquad = \sum_{n \in \mathbf{Z}} [n^4 + 6n^2 + 2\rho n^2 + \rho^2 + 2\rho + 1]\, I_n(t) \\
&\qquad = 3t^2 + 7t + 2\rho t + \rho^2 + 2\rho + 1 \leqslant 4(1 + t + \rho)^2
\end{aligned}$$

whence (5.7) follows by the Hölder inequality. The right-hand side of (5.2) will be evaluated by means of the following three lemmas.

Lemma 5.8. If $0 \leqslant \lambda \leqslant 3$, $\rho \geqslant 0$ and

$$\sum_{k=-n}^{n} |f_k| \leqslant p[n^2 + \rho]^{\lambda/2}$$

for $n \in \mathbb{N}$ then

$$\left| \sum_{n \in \mathbf{Z}} I_n(t)(f_n - f_{n-1}) \right| \leqslant \frac{32p}{1+t} (1 + t + \rho)^{\lambda/2}$$

Proof. Using (5.6) and

$$[(n+1)^2 + \rho]^b - [n^2 + \rho]^b \leqslant 2b[(n+1)^2 + \rho]^{b-1/2}$$

we obtain that

$$\begin{aligned}
\left| \sum_{n \in \mathbf{Z}} I_n(t)(f_n - f_{n-1}) \right| &= \left| \sum_{n \in \mathbf{Z}} (I_n - I_{n+1}) f_n \right| \\
&\leqslant \sum_{n=0}^{\infty} (I_n - I_{n+1})(|f_n| + |f_{-n-1}|) \\
&\leqslant \frac{4}{1+t} \sum_{n=0}^{\infty} (n+1) I_n(|f_n| + |f_{-n-1}|) \\
&\leqslant \frac{4}{1+t} \sum_{n=0}^{\infty} (n+1)(I_n - I_{n+1}) \sum_{k=-n-1}^{n+1} |f_k| \\
&\leqslant \frac{4p}{1+t} \sum_{n=0}^{\infty} [(n+1)^2 + \rho]^{\lambda/2 + 1/2} (I_n - I_{n+1}) \\
&\leqslant \frac{16p}{1+t} \sum_{n=0}^{\infty} [(n+1)^2 + \rho]^{\lambda/2} I_n
\end{aligned}$$

whence (5.8) follows by (5.7). ■

Lemma 5.9. If $0 \leqslant \lambda \leqslant 3$, $\rho \geqslant 0$ and

$$\sum_{k=-n}^{n} g_k^2 \leqslant q^2 (n^2 + \rho)^{\lambda/2}$$

for $n \in \mathbb{N}$ then

$$\left| \sum_{n \in \mathbf{Z}} I_n(t)(g_n - g_{n-1}) \right| \leqslant \frac{48q}{1+t} (1 + t + \rho)^{\lambda/4 + 1/4}$$

Proof. The first steps are the same as in the proof of Lemma 5.8; then by the Cauchy inequality we obtain

$$\left| \sum_{n\in\mathbf{Z}} I_n(t)(g_n - g_{n-1}) \right|$$
$$\leqslant \frac{8}{1+t} \sum_{n=0}^{\infty} (n+1)^{3/2}(I_n - I_{n+1}) \left[\sum_{k=-n-1}^{n+1} g_k^2 \right]^{1/2}$$
$$\leqslant \frac{8q}{1+t} \sum_{n=0}^{\infty} [(n+1)^2 + \rho]^{\lambda/4+3/4}(I_n - I_{n+1})$$
$$\leqslant \frac{24q}{1+t} \sum_{n=0}^{\infty} [(n+1)^2 + \rho]^{\lambda/4+1/4} I_n(t)$$

whence the statement follows by (5.7). ■

Lemma 5.10. Suppose that $\omega \in \Omega_w^\lambda$ with $0 < \lambda < 3$, then (2.7) implies that

$$\lim_{t\to+\infty} \sum_{n\in\mathbf{Z}} I_n(t)(\omega_n + \omega_{n-1}) = 2z - w$$

Proof. Let $s_n = \omega_{-n-1} + \omega_{-n} + \cdots + \omega_{n-1} + \omega_n$ and $S_n = s_0 + s_1 + \cdots + s_n$, then by (5.5) we have

$$\sum_{n\in\mathbf{Z}} I_n(t)(\omega_n + \omega_{n-1}) = I_0 s_0 + I_1 s_1 + \sum_{n=2}^{\infty} I_n(s_n - s_{n-2})$$
$$= \sum_{n=0}^{\infty} (I_n - I_{n+2})\, s_n = \frac{2}{t} \sum_{n=0}^{\infty} (n+1)\, I_{n+1} s_n$$
$$= \frac{2}{t} I_1 S_0 + \frac{2}{t} \sum_{n=1}^{\infty} (n+1)\, I_{n+1}(S_n - S_{n-1})$$
$$= \frac{2}{t} \sum_{n=1}^{\infty} [nI_n - (n+1)\, I_{n+1}]\, S_{n-1}$$

We introduce now the abbreviation

$$J_n(t) = \frac{n}{t}(n+1)[nI_n(t) - (n+1)\, I_{n+1}(t)] \tag{5.11}$$

for $n \in \mathbb{N}$ and $t > 0$. It is easy to check that $\lim J_n(t) = 0$ as $t \to +\infty$ for each $n \in \mathbb{N}$, and

$$\sum_{n=1}^{\infty} J_n(t) = 1 \tag{5.12}$$

for each $t \geqslant 0$; moreover,

$$\sum_{n=1}^{\infty} |J_n(t)| \leqslant \frac{1}{t} \sum_{n=1}^{\infty} n(n+1) I_{n+1} + \frac{1}{t} \sum_{n=1}^{\infty} n^2(n+1)(I_n - I_{n+1})$$
$$\leqslant \frac{1}{2} + \frac{1}{t} \sum_{n=1}^{\infty} 2n^2 I_n(t) = \frac{3}{2} \tag{5.13}$$

Therefore the summation kernel $J_n(t)$ transforms convergent sequences into their limit as $t \to +\infty$. Consequently, it is sufficient to show that $n^{-2}S_n$ converges to $z - w/2$ as n goes to infinity. However,

$$S_n = \sum_{k=0}^{n} \sum_{j=-k}^{k} \omega_j + \sum_{j=1}^{n+1} (\omega_{-j} + j\omega) - \frac{1}{2}(n+1)(n+2)w$$

and by the Cauchy inequality

$$\sum_{j=1}^{n+1} |\omega_{-j} + jw| \leqslant (n+1)^{1/2} \left[\sum_{k=-n-1}^{n+1} (\omega_k - kw)^2 \right]^{1/2}$$

thus (2.7) and $\omega \in \Omega_e^\lambda$ with $\lambda < 3$ imply the statement. ∎

Now we are in a position to prove the local stability of stationary solutions.

6. PROOF OF THEOREM 2.6

Let us rewrite (1.2) as

$$\dot{\delta}_k(t) = \frac{\sigma}{2} [\delta_{k-1}(t) + \delta_{k+1}(t) - 2\delta_k(t)] + g_k(t) - g_{k-1}(t)$$

where $\sigma = 2U''(w)$ and

$$g_k(t) = U'(\delta_{k+1}(t)) - U'(\delta_k(t)) - U''(w)[\delta_{k+1}(t) - \delta_k(t)]$$

then (5.2) results in

$$\delta_0(\mathbb{P}^t\omega) = \delta_0(\mathbb{P}_0^t\omega) + \int_0^t \sum_{n\in\mathbf{Z}} I_n(\sigma t - \sigma s)[g_n(s) - g_{n-1}(s)]\, ds \tag{6.1}$$

where

$$(\mathbb{P}_0^t\omega)_m = \sum_{n\in\mathbf{Z}} I_n(\sigma t)\, \omega_{m-n}, \qquad m \in \mathbf{Z} \tag{6.2}$$

denotes the solution of the associated linear approximation (1.5). Since $|g_k| \leqslant (1/c)|\delta_{k+1} - \delta_k|$ in view of (2.1), (4.19) implies a bound for $\delta_0(\mathbb{P}^t\omega) - \delta_0(\mathbb{P}_0^t\omega)$ via Lemma 5.9. Indeed, let $h = 1$ and $u = Z_1(t, x, \omega)$. Since (4.3) implies that $P(u - \theta, 0, r) = O(r^\lambda)$, the right-hand side of (4.19) is bounded by a multiple of $(r^2 + Mt)^{\lambda/2}$; thus we have a finite $q(\omega)$ such that

$$\sum_{k=-n}^{n} (\delta_{k+1}(\mathbb{P}^t\omega) - \delta_k(\mathbb{P}^t\omega))^2 \leqslant q^2(\omega)(1+t)^{-2}(n^2 + Mt)^{\lambda/2} \tag{6.3}$$

Therefore $g_k(s)$ satisfies the conditions of Lemma 5.9 with $q = K_1 \cdot q(\omega)/(1+s)$ and $\rho = \sigma s$; consequently,

$$\begin{aligned} |\delta_0(\mathbb{P}^t\omega) - \delta_0(\mathbb{P}_0^t\omega)| &\leqslant K_2 \int_0^t \frac{(1+t)^{\lambda/4+1/4}}{(1+t-s)(1+s)}\, ds \\ &\leqslant 2K_2(1+t)^{\lambda/4-3/4} \log(1+t) \end{aligned} \tag{6.4}$$

where K_2 depends only on ω; thus we have an estimate for the rate of convergence in Theorem 2.6.

Theorem 6.5. If $\lambda \leqslant 3$ and $\omega \in \Omega_w^\lambda$ then we have for each $m \in \mathbf{Z}$

$$\limsup_{t\to\infty} \frac{t^{3/4-\lambda/4}}{\log t} |\delta_m(\mathbb{P}^t\omega) - w| < +\infty$$

Proof. Applying Lemma 5.9 to

$$\delta_0(\mathbb{P}^t\omega) = \sum_{n\in\mathbf{Z}} I_n(\sigma t)(\omega_{n+1} - \omega_n)$$

we obtain that

$$|\delta_0(\mathbb{P}_0^t\omega) - w| \leqslant K_3(1+t)^{\lambda/4-3/4} \tag{6.6}$$

where K_3 depends only on ω; thus we have the statement for $m = 0$, whence the general case follows directly by (6.3). ■

Remark 6.7. If the initial distribution of points is a Poisson process of intensity $1/w$ then the second moment of $\delta_0(\mathbb{P}_0^t\omega) - w$ equals $w^2 I_0(2\sigma t) = O(t^{-1/2})$, which corresponds to $\lambda = 2$ here.

Remark 6.8. It has not been exploited in the proof that the linear approximation $\mathbb{P}_0^t$ is fitted at the value w of typical distances. Using (2.2) and a more sophisticated version of Lemma 5.9, the exponent $\lambda/4 - 3/4$ in (6.4) can be replaced by $\lambda/2 - 3/2$, i.e., $(\mathbb{P}_0^t\omega)$ approximates $\delta(\mathbb{P}^t\omega)$ better than its limit w; cf. (6.7).

7. PROOF OF THEOREM 2.9

Now we rewrite (1.1) as

$$\dot{\omega}_k(t) = \frac{\sigma}{2}\left[\omega_{k-1}(t) + \omega_{k+1}(t) - 2\omega_k(t)\right] + f_k(t) - f_{k-1}(t)$$

where $\sigma = 2U''(w)$ and

$$f_k(t) = U'(\delta_k(t)) - U'(w) - U''(w)(\delta_k(t) - w)$$

thus

$$(\mathbb{P}^t\omega)_0 = (\mathbb{P}_0^t\omega)_0 + \int_0^t \sum_{n\in\mathbf{Z}} I_n(\sigma t - \sigma s)[f_n(s) - f_{n-1}(s)]\, ds \quad (7.1)$$

Since $|f_k| \leqslant C\,|\delta_k - w|^2$ in view of (2.2), while (4.18) yields

$$\sum_{k=-n}^{n} [\delta_k(\mathbb{P}^t\omega) - w]^2 \leqslant \frac{p^2(\omega)}{1+t}(n^2 + Mt)^{\lambda/2} \quad (7.2)$$

there exists a constant K_4 depending only on ω such that $f_k(s)$ satisfies the conditions of Lemma 5.8 with $p = K_4/(1+s)$ and $\rho = \sigma s$; consequently,

$$\begin{aligned} |(\mathbb{P}^t\omega)_0 - (\mathbb{P}_0^t\omega)_0| &\leqslant K_5 \int_0^t \frac{(1+t)^{\lambda/2}\, ds}{(1+t-s)(1+s)} \\ &\leqslant 2K_5(1+t)^{\lambda/2-1}\log(1+t) \quad (7.3) \end{aligned}$$

On the other hand, Lemma 5.10 yields

$$\lim_{t\to\infty} [(\mathbb{P}_0^t\omega)_0 + (\mathbb{P}_0^t\omega)_{-1}] = 2z - w \quad (7.4)$$

while (7.2) holds for $\mathbb{P}_0^t$, as well; thus comparing (7.2), (7.3), and (7.4) we obtain Theorem 2.9. Conditions for the rate of convergence should be given in terms of the initial configuration.

Theorem 7.5. Let $\lambda < 2$, $\omega \in \Omega_w^\lambda$ and suppose (2.2) and

$$\sup_{n\in\mathbb{N}} n^{1-2\lambda} \sum_{k=1}^{n} \left[\left(\sum_{j=-k}^{k} \omega_j\right) - (2k+1)z\right]^2 < +\infty$$

then for each $m \in \mathbf{Z}$ we have

$$\limsup_{t\to\infty} \frac{t^{1-\lambda/2}}{\log t}\,|(\mathbb{P}^t\omega)_m - z - mw| < +\infty$$

Proof. Let $g_n = \omega_{-n} + \omega_{-n+1} + \cdots + \omega_n - (2n+1)z$ if $n > 0$, while $g_0 = \omega_0 - z$, $g_{-1} = z - \omega_0$ and $g_n = -g_{-n-1}$ if $n < 0$; then

$$\sum_{n \in \mathbf{Z}} I_n(\sigma t)\, \omega_n = \frac{1}{2} \sum_{n \in \mathbf{Z}} I_n(\sigma t)(\omega_n + \omega_{-n})$$

$$= z + \frac{1}{2} \sum_{n \in \mathbf{Z}} I_n(\sigma t)(g_n - g_{n-1})$$

thus Lemma 5.9 yields

$$|(\mathbb{P}_0^t \omega)_0 - z| \leqslant K_6 (1+t)^{\lambda/2 - 1} \tag{7.6}$$

with K_6 depending only on ω; thus (7.2) and (7.3) result in the statement. ■

8. PROOF OF THEOREM 3.4

First we reformulate the Riesz criterion of compactness in $\mathbb{L}_2$ for $\mathbb{H}_e$; let $\mathbf{T}_h u(x) = u(x+h)$ be as in Section 4.

Lemma 8.1. Let $E \subset \mathbb{H}_e$ and suppose for each $r \in \mathbb{N}$ that

$$\sup_{u \in E} \|u\|_r < +\infty$$

and

$$\lim_{\varepsilon \to 0} \sup_{u \in E} \|u - \mathbf{T}_\varepsilon u\|_r = 0$$

then E is precompact in $\mathbb{H}_e$.

Proof. In view of the Riesz criterion and the diagonal principle we can select a sequence $u_n \in E$ and a measurable $u_\infty : \mathbb{R} \to \mathbb{R}$ such that u_∞^2 is locally integrable and

$$\lim_{n \to \infty} \int_{-r}^{r} [u_n - u_\infty]^2 \, dx = 0 \tag{8.2}$$

for each $r \in \mathbb{N}$. On the other hand, as

$$\sum_{k=1}^{\infty} e^{-k/2} \int_{-kr}^{kr} u^2(x)\, dx \leqslant 2\, \|u\|_{2r}^2$$

for each $r \in \mathbb{N}$, we have

$$\sum_{k=m}^{\infty} e^{-k} \int_{-kr}^{kr} u^2(x)\, dx \leqslant 2 e^{-m/2}\, \|u\|_{2r}^2 \tag{8.3}$$

for all $m, r \in \mathbb{N}$. Comparing (8.2) and (8.3) we see that $u_\infty \in \mathbb{H}_e$ and $\lim u_n = u_\infty$ in $\mathbb{H}_e$, too. ■

To derive the second condition of Lemma 8.1 from Proposition 4.14 the following elementary property of step functions will be needed.

Lemma 8.4. Let $0 < h \leqslant 1$ and $0 < \varepsilon \leqslant 1$. If $u \in \mathbb{H}_e$ is constant on the intervals $[mh, mh + h)$, $m \in \mathbb{N}$; i.e., $u(x) = u(h[x/h])$, then for each $r \in \mathbb{N}$ we have

$$\|u - \mathbf{T}_\varepsilon u\|_r^2 \leqslant e\frac{\varepsilon}{h}\left(1 + \frac{\varepsilon}{h}\right)\|u - \mathbf{T}_h u\|_r^2$$

Proof. Let $m = [\varepsilon/h]$ and $s = \varepsilon - mh$. Since $0 \leqslant s < h$, for $k \in \mathbb{N}$

$$\int_{-kh}^{kh} [u(x+s) - u(x)]^2\, dx = \frac{s}{h}\int_{-kh}^{kh} [u(x+h) - u(x)]^2\, dx$$

Thus from

$$\begin{aligned} u(x+\varepsilon) - u(x) = u(x+\varepsilon) &- u(x+mh) + u(x+mh) \\ &- u(x+mh-h) + \cdots + u(x+h) - u(x) \end{aligned}$$

we obtain by the Cauchy inequality that

$$\begin{aligned} &\int_{-nr}^{nr} [u(x+\varepsilon) - u(x)]^2\, dx \\ &\quad\leqslant (m+1)\int_{-nr}^{nr} [u(x+\varepsilon) - u(x+mh)]^2\, dx \\ &\quad\quad + (m+1)\sum_{k=0}^{m-1}\int_{-nr}^{nr} [u(x+kh+h) - u(x+kh)]^2\, dx \\ &\quad\leqslant (m+1)\left(m + \frac{s}{h}\right)\int_{-nr-1}^{nr+1} [u(x+h) - u(x)]^2\, dx \end{aligned}$$

which completes the proof as $\varepsilon = mh + s$ and $h \leqslant 1 \leqslant r$. ■

Define now $y_h \in \mathbb{C}(\mathbb{R}_+, \mathbb{H}_e)$ as the unique solution of $\dot{y} = \nabla_h^* U'(\nabla_h y)$ with initial condition

$$y_h(0, x) = \int_{h[x/h]}^{h + h[x/h]} z_0(s)\, ds \tag{8.5}$$

and let $z_h(t,x) = h^{-1} y_h(t,x)$; we are going to show that the family z_h, $0 < h \leqslant 1$ is precompact in $\mathbb{C}(\mathbb{R}_+, \mathbb{H}_e)$. Indeed, as

$$\int_{-nr}^{nr} y_h^2(0,x)\,dx \leqslant h^2 \int_{-1-nr}^{1+nr} z_0^2(x)\,dx \tag{8.6}$$

i.e., $\|y_h(0,.)\|_r^2 \leqslant eh^2 \|z_0\|_r^2$; choosing $u = y_h$ and $\tilde{u} = 0$ in Proposition 4.16 we obtain by (4.3) that

$$\|z_h(t,.)\|_{\bar{r}}^2 + ch^{-2} \int_0^t \|\nabla_h y_h(s,.)\|_{\bar{r}}^2\,ds \leqslant 72\,\|z_0\|_{\bar{r}}^2 \tag{8.7}$$

where $\bar{r} = 1 + [(r^2 + Mt)^{1/2}]$. Similarly,

$$\begin{aligned}\int_{-nr}^{nr} [\nabla_h y_h(0,x)]^2\,dx &\leqslant h^2 \int_{-nr-1}^{nr+1} [\nabla_h z_0(x)]^2\,dx\\ &\leqslant h \int_{-nr-1}^{nr+1} \int_x^{x+h} [z_0'(s)]^2\,ds\,dx\\ &\leqslant h^2 \int_{-nr-2}^{nr+2} [z_0'(x)]^2\,dx\end{aligned} \tag{8.8}$$

i.e., $\|\nabla_h y_h(0,.)\|_r^2 \leqslant e^2 h^2 \|z_0'\|_r^2$; thus Proposition 4.16 yields

$$\|\nabla_h z_h(t,.)\|_{\bar{r}}^2 + c \int_0^t \|\dot{z}_h(s,.)\|_{\bar{r}}^2\,ds \leqslant 216\,\|z_0'\|_{\bar{r}}^2 \tag{8.9}$$

where $\bar{r}$ is the same as in (8.7). Since (8.7), (8.9), and Lemma 8.4 imply the conditions of Lemma 8.1, for any $T > 0$ there is a compact $E_T \subset \mathbb{H}_e$ such that $y_h(t,.) \in E_T$ if $0 \leqslant t \leqslant T$. On the other hand, as

$$\|z_h(t+s,.) - z_h(t,.)\|_r^2 \leqslant s \int_t^{t+s} \|\dot{z}_h(u,.)\|_r^2\,du \tag{8.10}$$

we see that the family $z_h(t,.)$, $0 < h \leqslant 1$ is equicontinuous on finite intervals of time, thus the Arzela–Ascoli theorem can be applied. We obtain that there exists a $z \in \mathbb{C}(\mathbb{R}_+, \mathbb{H}_e)$ and a sequence $h_n > 0$ such that $\lim h_n = 0$ and

$$\lim_{n\to\infty} \sup_{0\leqslant t\leqslant T} \|z_{h_n}(t,.) - z(t,.)\|_r = 0 \tag{8.11}$$

To identify z as the weak solution to $\dot{z} = \frac{1}{2} z''$, let $\varphi \in \mathbb{C}_0^2(\mathbb{R})$ and set $\varphi_h(x) = \varphi(h[x/h])$; then

$$\int_{-\infty}^{+\infty} \varphi_h(x)\, z_h(t,x)\, dx$$

$$= \int_{-\infty}^{+\infty} \varphi_h(x)\, z_0(x)\, dx + \int_0^t \int_{-\infty}^{+\infty} \varphi_h(x) \frac{1}{h} \nabla_h^* U'(\nabla_h y_h(s,x))\, dx\, ds$$

$$= \int_{-\infty}^{+\infty} \varphi_h(x)\, z_0(x)\, dx - \int_0^t \int_{-\infty}^{+\infty} [\nabla_h \varphi_h(x)] \frac{1}{h} U'(\nabla_h y_h(s,x))\, dx\, ds$$

$$= \int_{-\infty}^{+\infty} \varphi_h(x)\, z_0(x)\, dx + \frac{1}{2} \int_0^t \int_{-\infty}^{+\infty} [\Delta_h \varphi_h(x)]\, z_h(s,x)\, dx\, ds$$

$$+ \int_0^t \int_{-\infty}^{+\infty} [\nabla_h \varphi_h(x)] \frac{1}{h} a(\nabla_h y_h(s,x))\, dx\, ds \tag{8.12}$$

where $a(x) = x/2 - U'(x) = O(x^2)$ in view of (1.10). Therefore (8.7) and (8.11) imply

$$\int_{-\infty}^{+\infty} \varphi(x)\, z(t,x)\, dx = \int_{-\infty}^{+\infty} \varphi(x)\, z_0(x)\, dx$$

$$+ \frac{1}{2} \int_0^t \int_{-\infty}^{+\infty} \varphi''(x)\, z(s,x)\, dx\, ds \tag{8.13}$$

for any $\varphi \in \mathbb{C}_0^2(\mathbb{R})$. Since (8.13) determines $z \in \mathbb{C}(\mathbb{R}_+, \mathbb{H}_e)$ in a unique way, we have (8.11) for any sequence $h_n \to 0$. Finally, from (8.3) we see that $z_h(0,.)$ converges to z_0 in $\mathbb{H}_e$; thus using Proposition 4.14 with $u = Z_h$ and $\bar{u} = z_h$, and taking into account that

$$\int_{\Omega_e} \| Z_h(0,.,\omega) - z_h(0,.)\|_r^2\, \mu_h(d\omega) \to 0 \qquad \text{as} \quad h \to 0 \tag{8.14}$$

we obtain the statement by a direct calculation.

9. PROOF OF THEOREM 3.12

The main steps of the proof are essentially the same as above. Let $y_h \in \mathbb{C}(\mathbb{R}_+, \mathbb{H}_e)$ be defined as the unique solution to $\dot{y} = \nabla_h^* U'(\nabla_h y)$ with initial condition

$$y_h(0,x) = \frac{1}{h} \int_{h[x/h]}^{h[x/h]+h} y_0(s)\, ds \tag{9.1}$$

and put $w_h(t,x) = \nabla_h y_h(t,x)$. Since $\| y_h(0,.)\|_r^2 \leqslant e \| y_0 \|_r^2$ [cf. (8.6)], Proposition 4.14 yields by (4.3)

$$\| y_h(t,.)\|_r^2 \leqslant 72\, \| y_0 \|_r^2 \tag{9.2}$$

where $\bar{r} = 1 + [(r^2 + Mt)^{1/2}]$. Analogously as in (8.8) it follows that $\|w_h(0,.)\|_r^2 \leqslant e^2 \|y_0'\|_r^2$, thus Proposition 4.16 results in

$$\|w_h(0,.)\|_r^2 + c\int_0^t \|\dot{y}_h(s,.)\|_r^2\, ds \leqslant 216\, \|y_0'\|_{\bar{r}}^2 \tag{9.3}$$

with $\bar{r}$ as in (9.2). Finally, as

$$\begin{aligned}\|\nabla_h^* U'(w_h(0,.))\|_r^2 &\leqslant c^{-2}\, \|\nabla_h^* w_h(0,.)\|_r^2 \\ &\leqslant (e/c)^2\, \|\nabla_h^* y_0'\|_r^2 \leqslant (e^2/c^2)^2\, \|y_0''\|_r^2\end{aligned}$$

and

$$\|\nabla_h w_h(t,.)\|_r^2 \leqslant e\, \|\nabla_h^* w_h(t,.)\|_r^2 \leqslant 2ec^{-2} R_h(y_h, t, r)$$

Proposition 4.17 and (4.3) imply

$$\|\nabla_h w_h(t,.)\|_r^2 + \frac{1}{ec}\int_0^t \|\dot{w}_h(s,.)\|_r^2\, ds \leqslant 72(e/c)^4\, \|y_0''\|_{\bar{r}}^2 \tag{9.4}$$

with $\bar{r}$ as above. Therefore, taking into account

$$\|y_h(t+\varepsilon,.) - y_h(t,.)\|_r^2 \leqslant \varepsilon \int_t^{t+\varepsilon} \|\dot{y}_h(s,.)\|_r^2\, ds \tag{9.5}$$

and the analogous inequality for w_h, by Lemma 8.4 and Lemma 8.1 we obtain that both families y_h and w_h, $0 < h \leqslant 1$ are precompact in $\mathbb{C}(\mathbb{R}_+, \mathbb{H}_e)$, i.e., we can select a sequence $h_n > 0$ and $y, w \in \mathbb{C}(\mathbb{R}_+, \mathbb{H}_e)$ such that $h_n \to 0$ and for any $r \in \mathbb{N}$ we have

$$\lim_n \left[\|y_{h_n}(t,.) - y(t,.)\|_r + \|w_{h_n}(t,.) - w(t,.)\|_r\right] = 0 \tag{9.6}$$

uniformly in finite intervals of time. Since

$$\int_{-\infty}^{+\infty} (\nabla_h^* \varphi(x))\, y_h(t,x)\, dx = -\int_{-\infty}^{+\infty} \varphi(x)\, w_h(t,x)\, dx$$

is an identity, we have

$$\int_{-\infty}^{+\infty} \varphi'(x)\, y(t,x)\, dx = -\int_{-\infty}^{+\infty} \varphi(x)\, w(t,x)\, dx \tag{9.7}$$

for $\varphi \in \mathbb{C}_0^1(\mathbb{R})$, consequently $y(t, .)$ is absolutely continuous and $y'(t, x) = w(t, x)$, i.e., $y \in \mathbb{C}(\mathbb{R}_+, \mathbb{H}_e^1)$. Thus from (9.6) we obtain that y and w satisfy (2.11) and (2.9) for $\varphi \in \mathbb{C}_0^1(\mathbb{R})$ and for $\varphi \in \mathbb{C}_0^2(\mathbb{R})$, respectively. From (9.4) by Lemma 8.4 the weak differentiability of $w(t, .)$ follows for all $t > 0$, thus we have $y(t, .) \in \mathbb{H}_e^2$, too.

The next step is to prove uniqueness of the Cauchy problem for (3.11) in the class $\mathbb{C}(\mathbb{R}_+, \mathbb{H}_e^1)$. Observe that (3.11) extends to functions $\varphi(x) = f(x, r)\, u(x)$ if $u \in \mathbb{H}_e^1$ and f denotes the cut-off function of Section 4. Therefore, if $\varepsilon > 0$,

$$\int f(x, r)\, u(x)[y(t+\varepsilon, x) - y(t, x)]\, dx$$

$$= -\int_t^{t+\varepsilon} \int [(\nabla f(x, r))\, u(x) + f(x, r)\, u'(x)]\, U'(w(s, x))\, dx\, ds \tag{9.8}$$

where $w(t, x) = y'(t, x)$. Since $|\nabla f| \leqslant f$, choosing $u(x) = y(t+\varepsilon, x) - y(t, x)$ an easy calculation yields

$$\begin{aligned} &\| y(t+\varepsilon, .) - y(t, .)\|_r^2 \\ &\quad \leqslant C_1 \| y(t+\varepsilon, .) - y(t, .)\|_r \int_t^{t+\varepsilon} \| w(s, .)\|_r\, ds \\ &\qquad + C_1 \| w(t+\varepsilon, .) - w(t, .)\|_r \int_t^{t+\varepsilon} \| w(s, .)\|_r\, ds \end{aligned} \tag{9.9}$$

Suppose now that $\bar{y} \in \mathbb{C}(\mathbb{R}_+, \mathbb{H}_e^1)$ is another weak solution with $\bar{y}(0, .) = y(0, .)$, and let $v = y - \bar{y}$, $\bar{w} = \bar{y}'$, $t_k = tk/m$, then

$$\begin{aligned} &\int f(x, r)\, v^2(t, x)\, dx \\ &\quad = -\sum_{k=0}^{m-1} \int f(x, r)[v(t_{k+1}, x) - v(t_k, x)]^2\, dx \\ &\qquad + 2 \sum_{k=0}^{m-1} \int f(x, r)\, v(t_{k+1}, x)[v(t_{k+1}, x) - v(t_k, x)]\, dx \end{aligned} \tag{9.10}$$

is an identity. In view of (9.9), v has a vanishing quadratic variation, as a trajectory in $\mathbb{H}_e$, thus putting $B(s, x) = U'(w(s, x)) - U'(\bar{w}(s, x))$ and letting $m \to +\infty$ we obtain that

$$\begin{aligned}
&\int f(x,r)\, v^2(t,x)\, dx \\
&\quad = -2\int_0^t \int (\nabla f(x,r))\, v(s,x)\, B(s,x)\, dx\, ds \\
&\quad\quad -2\int_0^t \int f(x,r)\, v'(s,x)\, B(s,x)\, dx\, ds \\
&\quad \leqslant 2\int_0^t \int f\cdot v\cdot B\, dx\, ds - 2c\int_0^t \int f\cdot B^2\, dx\, ds \\
&\quad \leqslant \frac{1}{2\cdot c}\int_0^t \int f(x,r)\, v^2(s,x)\, dx\, ds
\end{aligned}$$

whence $y = \bar{y}$ follows by the Gronwall lemma.

Now we are in a position to complete the proof of Theorem 3.12. Since $y_h(0,.)$ converges to y_0 in $\mathbb{H}_e$ as h goes to zero, we have

$$\lim_{h\to 0} \int_{\Omega_e} \| Y_h(0,.,\omega) - y_h(0,.)\|_r^2\, \mu_h(d\omega) = 0$$

for each $r \in \mathbb{N}$, furthermore (9.6) holds for any sequence $h_n \to 0$, thus Proposition 4.14 implies the last assertion we have to prove.

10. PROOF OF THEOREM 3.18

First we show that the additional conditions of Theorem 3.18 hold for all $t > 0$. For this purpose we need a lower a priori bound for the distance of particles. Consider

$$J_h(t,\varphi,\omega) = \int_{-\infty}^{+\infty} \varphi(x)\, W_h(t,x,\omega)\, dx \tag{10.1}$$

and suppose that $\varphi \in \mathbb{C}_0^2(\mathbb{R})$ is nonnegative and

$$\Delta_s \varphi(x) \geqslant -K\varphi(x) \qquad \text{for} \quad 0 < s \leqslant h \tag{10.2}$$

with some $K > 0$, then

$$\begin{aligned}
\dot{J}_h(t,\varphi,\omega) &= \int_{-\infty}^{+\infty} \varphi(x)\, \Delta_h U'(W_h(t,x,\omega))\, dx \\
&= \int_{-\infty}^{+\infty} [\Delta_h \varphi(x)]\, U'(W_h(t,x,\omega))\, dx \geqslant -\frac{K}{c} J_h(t,\varphi,\omega)
\end{aligned}$$

as $W_h \geqslant 0$; consequently,

$$J_h(t, \varphi, \omega) \geqslant J_h(0, \varphi, \omega) \exp(-Kt/c) \tag{10.3}$$

Letting h go to zero in (10.3) we see that $y(t, .)$ is strictly increasing if y_0 is so. Indeed, if $y(t, a) = y(t, b)$ then $w(t, x) = 0$ for $a < x < b$, thus choosing φ such that (10.2) holds with some $h > 0$, and $\varphi(x) = 0$ if $x < a$ or $x > b$, the contradiction $y_0(a) = y_0(b)$ is obtained. Therefore $\rho(t, x)$ is well defined.

Of course, strict monotonicity of $y(t, .)$ does not exclude $y'(t, x) = 0$ on a nowhere dense set, i.e., $\rho(t, y) = +\infty$ is possible. We show, however, that $\rho(t, .)$ is locally integrable; consequently $N(t, \varphi)$ is well defined. First we have to extend the additional condition of Theorem 3.18 to positive values of time. To prove

$$\lim_{r\to\infty} \sup_{h>0} \sup_{t\leqslant T} \mu_h[Y_h(t, -r, .) < -b, Y_h(t, r, .) > b] = 1 \tag{10.4}$$

observe that (10.3) implies

$$\lim_{r\to\infty} \sup_{h>0} \sup_{t\leqslant T} \mu_h[Y_h(t, r, .) - Y_h(t, 0, .) > b] = 1 \tag{10.5}$$

and the tail of $Y_h(t, 0, .) - Y_h(t, -r, .)$ can be estimated in the same way. On the other hand, as Y_h increases with x, we have

$$\int_{-1}^{0} Y_h(t, x, \omega)\, dx \leqslant Y_h(t, 0, \omega) \leqslant \int_0^1 Y_h(t, x, \omega)\, dx \tag{10.6}$$

thus (10.5) and Proposition 4.14 imply (10.4). Consequently, for any $\varphi: \mathbb{R} \to \mathbb{R}$ with a compact support we have

$$\lim_{r\to+\infty} \sup_{h>0} \mu_h \left[\int_{|x|>r} \varphi(Y_h(t, x, .)\, dx \neq 0\right] = 0$$

while Theorem 3.12 yields

$$\lim_{h\to 0} \mu_h \left[\int_{-r}^{r} |\varphi(Y_h(t, x, .) - \varphi(y(t, x))|\, dx > \varepsilon\right] = 0$$

for each $r > 0$, $\varepsilon > 0$ and uniformly continuous φ. This means that $N_h(t, \varphi, .)$ and $N(t, \varphi)$ are well defined, and $N_h(t, \varphi, .) \to N(t, \varphi)$ in probability as $h \to 0$.

ACKNOWLEDGMENT

An essential part of this work was done during the author's visit at I.H.E.S. I wish to express my thanks to Professor N. H. Kuiper for kind hospitality.

REFERENCES

1. F. Spitzer, Random processes defined through the interaction of an infinite particle system, in *Lecture Notes in Mathematics*, Vol. 89 (Springer Verlag, Berlin, 1969), pp. 201–223.
2. A. Garcia and H. Kesten, Unpublished; see the footnote on p. 221 in Ref. 1.
3. R. L. Dobrushnin and J. Fritz, Non-equilibrium dynamics of one-dimensional infinite particle systems with a hard-core interaction, *Commun. Math. Phys.* **55**:275–292 (1977).
4. R. Lang, On the asymptotic behaviour of infinite gradient systems, *Commun. Math. Phys.* **65**:129–149 (1979).
5. K. H. Fichtner and W. Freudenberg, Asymptotic behaviour of time evolutions of infinite particle systems, *Z. Wahrsch. Verw. Gebiete* **54**:141–159 (1980).
6. J. Fritz, Local stability and hydrodynamical limit of Spitzer's one-dimensional lattice model, *Commun. Math. Phys.* **86**:363–373 (1982).
7. C. Boldrighini, R. L. Dobrushnin, and Yu. M. Suhov, One-dimensional hard-rod caricature of hydrodynamics, *J. Stat. Phys.* **31**:577 (1983).
8. H. Rost, Hydrodynamik gekoppelter Diffusionen: Fluktuationen im Gleichgewicht, preprint, Heidelberg University, 1982.
9. C. Boldrighini, A. Pellegrinotti, and L. Triolo, Convergence to stationary states for infinite harmonic systems, *J. Stat. Phys.* **30**:123–155 (1983).
10. J. L. Daleckii and M. G. Krein, *Stability of solutions to differential equations in Banach spaces* (Nauka, Moscow, 1970) (in Russian).
11. J. L. Lions, Quelques methodes de résolution des problèmes aux limites non linéaries (Dunod, Paris, 1969).
12. T. Kato, *Perturbation theory for linear operators* (Springer Verlag, Berlin, 1980).
13. I. S. Gradshtein and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic Press, New York, 1980).
14. A. De Masi, N. Ianiro, A. Pellegrinotti, and E. Presutti, A survey of the hydrodynamical behaviour of many particle systems, in *Nonequilibrium Processes*, Vol. 11, J. L. Lebowitz, E. W. Montroll, eds., *Studies in Statistical Mechanics* (North-Holland, Amsterdam, 1984).
15. E. Scacciatelli, private communication.
16. E. Presutti and E. Scacciatelli, Time evolution of a one-dimensional point system: A note on Fritz's paper, *J. Stat. Phys.* **38**:647–653.